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Non-perturbative renormalization of Φ-derivable approximations in theories with multiple fields

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Abstract. We provide a renormalization procedure for Φ-derivable approximations in theories coupling different types of fields. We illustrate our approach on a scalar φ^4 theory coupled to fermions via a Yukawa-like interaction. The non-perturbative renormalization amounts to fixing the scalar coupling via a set of nested Bethe-Salpeter equations coupling fermions to scalars.

PACS. 11.10.Wx Finite-temperature field theory – 11.15.Tk Other nonperturbative techniques – 11.10.Gh Renormalization

1 Introduction

The two-particle-irreducible (2PI) effective action [1] has recently regained interest in many fields of research such as the QCD phase diagram [2] or the dynamics of quantum fields out of equilibrium [3]. There are however two major difficulties related to how fundamental properties such as UV finiteness or gauge invariance manifest themselves within a given truncation of the 2PI effective action. Recent progresses have been made concerning renormalization in the case of scalar theories [4]. Aside from the question of gauge invariance, an important step towards the understanding of renormalization in the case of gauge theories consists in considering a theory coupling multiple fields.

As an illustration, we consider here a massless fermionic field ψ at finite temperature, coupled to a massless self-interacting scalar field φ ($\lambda\varphi^4$ theory) via a Yukawa-like interaction $g\bar{\psi}\psi\varphi$. We concentrate on the symmetric phase for which it is enough to consider the 2PI effective action as a functional of the full propagators S and D:

$$\Gamma_{\rm 2PI}[S,D] = -\sum\!\!\!\!\!\!\int \Big[\ln S^{-1} - \Sigma S\Big] + \frac{1}{2}\sum\!\!\!\!\!\!\int \Big[\ln D^{-1} - \Pi D\Big] + \Phi[S,D]\,, \tag{1}$$

where Φ is the sum of 0-leg skeletons (2PI diagrams). When evaluated at its stationary point, the 2PI effective action provides a powerful way to compute the pressure of the system. Further thermodynamic quantities are obtained by taking derivatives. The propagators are given by the stationary condition on $\Gamma_{\rm 2PI}[S,D]$ which is suitably translated into a set of equations of motion for the self-energies Σ and Π. As an illustration we consider here the one-loop approximation:

$$\begin{aligned}\Sigma(P) &= -g^2\sum\!\!\!\!\!\!\int_Q D(Q)S(Q+P) - \not{P}\delta Z_\psi\,,\\ \Pi(K) &= \frac{1}{2}\,(\lambda+\delta\lambda)\sum\!\!\!\!\!\!\int_Q D(Q)\\ &\quad + g^2{\rm tr}\sum\!\!\!\!\!\!\int_Q S(Q)S(Q+K) + K^2\delta Z_\varphi\,.\end{aligned} \tag{2}$$

The sum integrals in the above equations are UV divergent. Renormalization consists in building temperature-independent counterterms δZ_φ, δZ_ψ and $\delta\lambda$ which absorb all the divergences. In dimensional regularization there is no need for mass counterterms. At one-loop level, the Yukawa coupling is not renormalized and we thus take $\delta g = 0$. A general discussion to higher loop orders may be found in [5].

2 Renormalization

One starts focusing on the zero-temperature solutions to the equations of motion that we denote by $S_{T=0}$ and $D_{T=0}$. The related self-energies are denoted by $\Sigma_{T=0}$ and $\Pi_{T=0}$ and determine the field strength counterterms δZ_ψ and δZ_φ through the renormalization conditions $\mathrm{d}\Sigma_{T=0}/\mathrm{d}\not{P}|_{P_*^2=-\mu^2} = 0$ and $\mathrm{d}\Pi_{T=0}/\mathrm{d}K^2|_{K_*^2=-\mu^2} = 0$.

The next step consists in removing coupling singularities from the temperature-dependent contributions $\Sigma_T = \Sigma - \Sigma_{T=0}$ and $\Pi_T = \Pi - \Pi_{T=0}$. It is easy to check that

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the equation for Σ_T does not contain any UV divergence. In contrast the equation for Π_T contains logarithmic divergences. A diagrammatic analysis [5] reveals that these are exactly the same than those encoded in the four-point function with four scalar legs ($\Gamma_{\varphi\varphi}$). Renormalization of these divergences thus amounts to imposing the renormalization condition $\Gamma_{\varphi\varphi}(P_*, K_*) = \lambda$. To proceed, we thus need to know how to build $\Gamma_{\varphi\varphi}$, show that it can be renormalized and check that its renormalization removes the coupling divergences in Π_T.

The function $\Gamma_{\varphi\varphi}$ is built in two steps from a set of nested Bethe-Salpeter equations. These equations involve 2PI *kernels* which are directly related to derivatives of the functional Φ namely $\Lambda_{\psi\psi} = -\delta^2\Phi/\delta S\delta S$, $\Lambda_{\psi\varphi} = -2\delta^2\Phi/\delta S\delta D$, $\Lambda_{\varphi\psi} = -2\delta^2\Phi/\delta D\delta S^{\mathrm{t}}$ and $\Lambda_{\varphi\varphi} = 4\delta^2\Phi/\delta D\delta D$. In the first step, one builds up the four-point function with four fermionic legs $\Gamma_{\psi\psi}$ from $\Lambda_{\psi\psi}$:

$$\Gamma_{\psi\psi}(P,K)=\Lambda_{\psi\psi}(P,K)-\int_Q \Lambda_{\psi\psi}(P,Q)M(Q)\Gamma_{\psi\psi}(Q,K). \tag{3}$$

$\Gamma_{\psi\psi}$ is then combined with the 2PI kernels in order to generate a new kernel:

$$\begin{aligned}&\tilde{\Lambda}_{\varphi\varphi}(P,K) = \\ &\Lambda_{\varphi\varphi}(P,K) + \int_Q \Lambda_{\varphi\psi}(P,Q)M(Q)\Lambda_{\psi\varphi}(Q,K) \\ &-\int_Q\int_R \Lambda_{\varphi\psi}(P,Q)M(Q)\Gamma_{\psi\psi}(Q,R)M(R)\Lambda_{\psi\varphi}(R,K).\end{aligned} \tag{4}$$

The explicit distribution of fermionic indices in eqs. (3) and (4) can be found in [5] where we also define M as $M_{(\alpha\beta),(\gamma\delta)}(Q) = S_{\alpha\gamma}(Q)S_{\delta\beta}(Q)$. In the second step, a second Bethe-Salpeter equation builds up $\Gamma_{\varphi\varphi}$ from $\tilde{\Lambda}_{\varphi\varphi}$:

$$\Gamma_{\varphi\varphi}(P,K)=\tilde{\Lambda}_{\varphi\varphi}(P,K)-\int_Q \tilde{\Lambda}_{\varphi\varphi}(P,Q)D^2(Q)\Gamma_{\varphi\varphi}(Q,K). \tag{5}$$

At one loop level, it is simple to check that eq. (3) is finite. In contrast, eq. (4) contains logarithmic divergences. However, these are overall divergences from which it follows that differences such that $\tilde{\Lambda}_{\varphi\varphi}(P,Q) - \tilde{\Lambda}_{\varphi\varphi}(K,Q)$ are finite. Furthermore, by construction, the kernel $\tilde{\Lambda}_{\varphi\varphi}$ is 2PI with respect to scalar lines implying that $\tilde{\Lambda}_{\varphi\varphi}(P,Q) - \tilde{\Lambda}_{\varphi\varphi}(K,Q) \sim 1/Q$ at large Q and fixed P and K [5]. It is then possible to rewrite the equation for $\Gamma_{\varphi\varphi}$ in a UV finite form:

$$\begin{aligned}&\Gamma_{\varphi\varphi}(P,K) - \Gamma_{\varphi\varphi}(P_*,K_*) = \tilde{\Lambda}_{\varphi\varphi}(P,K) - \tilde{\Lambda}_{\varphi\varphi}(P_*,K_*) \\ &-\frac{1}{2}\int_Q \Big\{\tilde{\Lambda}_{\varphi\varphi}(P,Q) - \tilde{\Lambda}_{\varphi\varphi}(P_*,Q)\Big\}D^2(Q)\Gamma_{\varphi\varphi}(Q,K) \\ &-\frac{1}{2}\int_Q \Gamma_{\varphi\varphi}(P_*,Q)D^2(Q)\Big\{\tilde{\Lambda}_{\varphi\varphi}(Q,K)-\tilde{\Lambda}_{\varphi\varphi}(Q,K_*)\Big\}.\end{aligned} \tag{6}$$

Using that $\Gamma_{\varphi\varphi}(Q,K) \sim \log Q$, $D(Q) \sim 1/Q^2$ and $\tilde{\Lambda}_{\varphi\varphi}(Q,K) - \tilde{\Lambda}_{\varphi\varphi}(Q,K_*) \sim 1/Q$ at large Q, one checks that the integrals in (6) are UV finite. Thus $\Gamma_{\varphi\varphi}$ is finite, as announced.

Finally, in order to show that renormalization of $\Gamma_{\varphi\varphi}$ simultaneously removes the coupling divergences in the equation for Π_T, it is convenient to express the latter in terms of $\tilde{\Lambda}_{\varphi\varphi}$. Followings the steps presented in [5], one obtains:

$$\begin{aligned}\Pi_T(K) = &\frac{1}{2}\int_Q \tilde{\Lambda}_{\varphi\varphi}(K,Q)\delta D(Q) \\ &+\frac{1}{2}\int_{\tilde{Q}} \tilde{\Lambda}_{\varphi\varphi}(K,\tilde{Q})\sigma_\varphi(\tilde{Q}) + \Pi_r(K),\end{aligned} \tag{7}$$

where we have performed the Matsubara sum in order to separate implicit and explicit thermal dependences. The first integral in (7) involves $\delta D(Q) = D(Q) - D_{T=0}(Q) = -\Pi_T(Q)D^2(Q) + D_r(Q)$ with $D_r(Q) \sim 1/Q^6$ at large Q. The second integral involves $\sigma_\varphi(\tilde{Q}) = \epsilon(q_0)n(|q_0|)\rho_\varphi(q_0,q)$, a particular combination of the sign function $\epsilon(q_0)$, the scalar thermal factor $n(|q_0|)$ and the scalar spectral density $\rho_\varphi(q_0,q)$. $\tilde{Q}$ designates integration along the real axis in contrast to Q which designates integration along the imaginary axis. Finally $\Pi_r(K)$ is a finite function which decreases as $\sim 1/K^2$ at large K. Using eqs. (5) and (7) one shows that

$$\begin{aligned}\Pi_T(K) = &\Pi_r(K) + \frac{1}{2}\int_{\tilde{Q}} \Gamma_{\varphi\varphi}(K,\tilde{Q})\sigma_\varphi(\tilde{Q}) \\ &+\frac{1}{2}\int_Q \Gamma_{\varphi\varphi}(K,Q)\big\{D_r(Q)-D^2(Q)\Pi_r(Q)\big\}.\end{aligned} \tag{8}$$

Using the finiteness of $\Gamma_{\varphi\varphi}$ and the asymptotic properties of $\Gamma_{\varphi\varphi}$, σ_φ, D_r and Π_r, it is simple to check that this last equation is finite, as expected.

3 Conclusions

We have shown on a particular example how to implement renormalization of Φ-derivable approximations in theories with multiple fields (see also [5]). This work is particularly important for gauge theories where one needs to disentangle the UV divergences related to gauge, matter and ghost fields.

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